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The train marshalling problem

Elias Dahlhaus[a], Peter Horak[b,*], Mirka Miller[c], Joseph F. Ryan[d]

[a] *Department of Computer Science, University of Cologne, D-50969 Cologne, Germany*
[b] *Department of Mathematics, Kuwait University, Kuwait*
[c] *Department of Computer Science, University of Newcastle, NSW 2308, Australia*
[d] *Department of Mathematics, University of Newcastle, NSW 2308, Australia*

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Abstract

The problem considered in this paper arose in connection with the rearrangement of railroad cars in China. In terms of sequences the problem reads as follows: Train Marshalling Problem: Given a partition S of $\{1,\dots,n\}$ into disjoint sets $S_1,\dots,S_t$, find the smallest number $k=K(S)$ so that there exists a permutation $p(1),\dots,p(t)$ of $\{1,\dots,t\}$ with the property: The sequence of numbers $1,2,\dots,n,1,2,\dots,n,\dots,1,2,\dots,n$ where the interval $1,2,\dots,n$ is repeated k times contains all the elements of $S_{p(1)}$, then all elements of $S_{p(2)},\dots$, etc., and finally all elements of $S_{p(t)}$. The aim of this paper is to show that the decision problem: "Given numbers n,k and a partition S of $\{1,2,\dots,n\}$, is $K(S)\leqslant k$?" is NP-complete. In light of this, we give a general upper bound for $K(S)$ in terms of n. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

In this paper we consider a real-life problem dealing with rearranging cars of trains. The reader interested in the history of the problem is referred to [5]. Let T be a train with n cars, $a_1,a_2,\dots,a_n$ arriving at a station in the given order. The cars have different destinations and at the station we want to rearrange their order so that all cars with the same destination will be grouped together. To be able to rearrange the order of cars, the train is taken to a shunting yard where the rail splits into k "auxiliary" rails. The first car a_1 is taken to any of the k rails, where it will be the first car on that rail. In general, a car a_i can be taken to any of the k rails where it is placed behind the cars

* Corresponding author.

E-mail addresses: dahlhaus@informatik.uni-koeln.de (E. Dahlhaus), horak@math-1.sci.kuniv.edu.kw (P. Horak), mirka@cs.newcastle.edu.au (M. Miller), joe@math.newcastle.edu.au (J.F. Ryan)

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already there. At the end of the process all the cars from one of the k rails are placed at the beginning of the rearranged train, followed by all the cars from another rail, etc.

Since, in general, there may be several trains at the station to be processed at the same time, the target is to use as few auxiliary rails as possible for each train. Clearly, the number of rails needed for each train is never more than the number of destinations. We will denote the minimum number of auxiliary rails needed for the required rearrangement by $K(T)$. Trivially, $K(T)$ is at most the total number of destinations in T.

Knuth [4] suggested the following (mathematical) reformulation of the problem which is more easily understood. Although it is not difficult to see we show here in detail the equivalence of the two problems. Let $S=\{S_1,\dots,S_t\}$ be a partition of the set $I_n=\{1,2,\dots,n\}$. The numbers from I_n correspond to cars of a train, while elements of S correspond to destinations. Thus, cars a_i,a_j have the same destination if and only if the numbers i,j belong to the same part of S. Now, the Train Marshalling Problem (TMP) reads as follows: Find the smallest number $k=K(S)$ so that there is a permutation $\pi(1),\dots,\pi(t)$ of $1,\dots,t$ so that the sequence of numbers

$$1,2,\dots,n,1,2,\dots,n,\dots,1,2,\dots,n,$$

where the interval $1,2,\dots,n$ is repeated k times, contains all the elements from $S_{\pi(1)}$, followed by all the elements of $S_{\pi(2)},\dots,$ and finally all the elements of $S_{\pi(t)}$. In this formulation the trivial bound becomes $K(S)\leqslant t$.

Example. For $n=11,\ t=5$ and a partition

$$S=\{S_1,S_2,S_3,S_4,S_5,\}=\{\{1,6,11\},\{2,7\},\{3,8\},\{4,9\},\ \{5,10\}\},$$

we get $K(S)=4$, with the arrangement

$$\underbrace{\mathbf{1}\,2\,3\,4\,5\,\mathbf{6}\,7\,8\,9\,10\,\mathbf{11}}_{S_1}\,1\,\underbrace{\mathbf{2}\,3\,4\,5\,6\,\mathbf{7}}_{S_2}\underbrace{\mathbf{8}\,9\,10\,11\,1\,2\,\mathbf{3}}_{S_3}\underbrace{\mathbf{4}\,5\,6\,7\,8\,\mathbf{9}}_{S_4}\underbrace{\mathbf{10}\,11\,1\,2\,3\,4\,\mathbf{5}}_{S_5}\,6\dots 11.$$

This example is shown schematically as a rearrangement of railway cars in Figs. 1–3.

By *arrangement* we will understand any order of cars in a train in which all cars of the same destination are consecutive. The numbers placed into the first interval $1,\dots,n$ are indices of cars placed on the same auxiliary rail and then placed at the beginning of the train, the numbers from the second interval $1,2,\dots,n$ are indices of cars from the rail taken after cars from the first rail, etc.

We will abuse language slightly and sometimes instead of an “element” of S we will simply say a “destination”, and instead of “numbers” we will speak about “cars”. We will also use the term “round” for the interval $1,2,\dots,n$.

Obviously, if we have an arrangement of cars of a train S into $K(S)$ rounds, that is, an optimal arrangement, then the cars of the same destination D are either in the same round (all cars are on the same auxiliary rail) and their numbers are in the increasing order in the arrangement, or the cars are in two rounds (on two auxiliary rails). In the

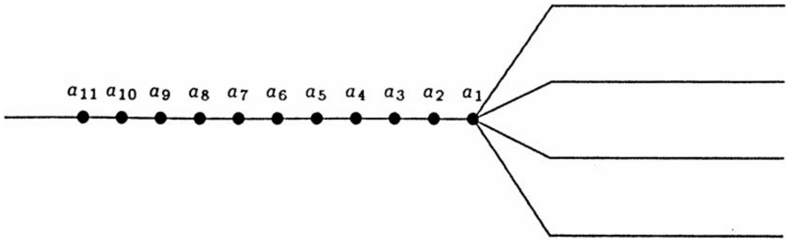

Fig. 1.

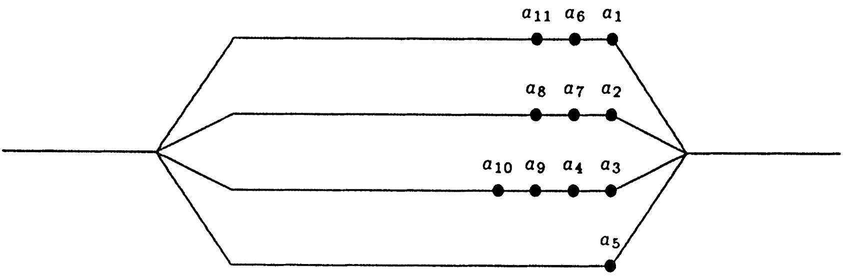

Fig. 2.

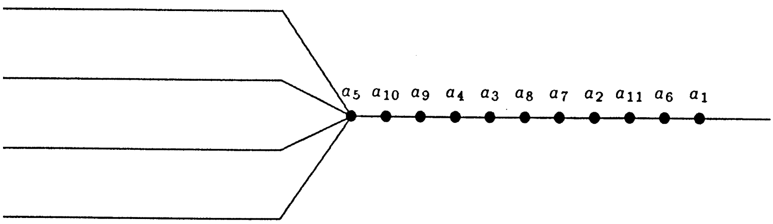

Fig. 3.

latter case their numbers are placed in the order $c_1 < \cdots < c_k$ (c_k being the last car in that round = auxiliary rail) followed by $d_1 < \cdots < d_s$, where $d_s < c_1$, and d_1 is the first car in the following round. In other words, if we know the number of the car from D which is placed as the first in the arrangement then we know order of all cars of D in this arrangement.

In [5] it has been shown that the value of $K(S)$ can be determined efficiently in some special cases.

Knuth [4] asked whether the following general decision problem is NP-complete.

Instance: Given natural numbers k, n and a partition S of the set I_n.

Question: Is $K(S) \leqslant k$?

In this paper we answer the Knuth's conjecture in the affirmative. In light of this result we give an upper bound on $K(S)$ in terms of n.

The above mentioned question might be seen as a part of the vast area of sorting problems with queues and stacks. The interested reader is referred to the monograph [1].

2. The general train marshalling problem is NP-complete.

The proof of the NP-completeness of our problem TMP is similar to the NP-completeness proof of the Minimum-Contraction Hypergraph Embedding in a Cycle, see [2].

In both cases the reduction is done from Numerical Matching with Target Sums (NMTS).

Instance: positive integers $a_1,\dots,a_n,b_1,\dots,b_n,c_1,\dots,c_n$.

Question: Does there exist a numerical matching, i.e., are there permutations π_1 and π_2 such that for all $i=1,\dots,n$, $a_{\pi_1(i)}+b_{\pi_2(i)}=c_i$?

The NP-completeness of NMTS is stated in [3, problem SP17].

Clearly, the NMTS remains NP-complete if we confine ourselves to instances satisfying the condition

$$\sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i = \sum_{i=1}^{n} c_i.$$

In what follows, we need additional conditions $a_i \geqslant 2, b_i > 2, c_j > b_i, c_j \geqslant a_i + 2$ for $1 \leqslant i,j \leqslant n$. To see that NMTS retains the property of being NP-complete also with these restrictions consider an instance $I:\ a_1,\dots,a_n,b_1,\dots,b_n,c_1,\dots,c_n$, and an instance $I^*:\ a_1+A,\dots,a_n+A,b_1+B,\dots,b_n+B,c_1+A+B,\dots,c_n+A+B$, where A and B are non-negative constants. Trivially, the instance I has a numerical matching iff the instance I^* does. To finish the proof it suffices to note that if we set $A=\max_{1\leqslant i\leqslant n} b_i, B=2+\max_{1\leqslant i\leqslant n} a_i$, then I^* satisfies all four required conditions.

2.1. The reduction

Let $a_1,\dots,a_n,b_1,\dots,b_n,c_1,\dots,c_n$ be positive integers, $a_i \geqslant 2, b_i > 2$, $c_j > b_i, c_j \geqslant a_i+2$ for $1 \leqslant i,j \leqslant n$. To the given numbers we will assign a partition ($=$ a train) R of the set $I_N=\{1,2,\dots,N\}, N=n\sum_{i=1}^{n} c_i$. The partition will be made up of $2n$ elements ($=$ destinations), $S_1,\dots,S_n,T_1,\dots,T_n$ corresponding to the numbers $a_1,\dots,a_n,b_1,\dots,b_n$, respectively. It would be very clumsy, for each number of I_N, to define by a formula to which element (destination) of R it belongs. Instead of it we will build our train R by constructing subblocks and blocks of cars with specified destinations. Our train will consists of n blocks $X_1,\dots,X_n$ of cars, and the blocks are placed in increasing order with respect to the index i, i.e., in the order $X_1X_2\dots X_n$. Each block X_i consists of c_i subblocks D^i_j, $j=1,\dots,c_i$, and the subblocks are again placed in increasing order with respect to the index j, i.e., $X_i=D^i_1D^i_2\dots D^i_{c_i}$. For any subblock D_j^i of R and any destination M there is in D_j^i either no car of the destination M or exactly one car of M.

For $i=1,\ldots,n$, there is a car of the destination S_k in the subblock D_j^i iff $j=c_i-a_k+1,\ldots,c_i$.

Further, for $i=1,\ldots,n$, there is a car of the destination T_k in the subblock D_j^i iff $j=1,\ldots,b_k$.

Inside any subblock all cars from S destinations, if any, precede cars from T destinations, and a car with destination $S_i(T_i)$ precedes a car with destination $S_j(T_j)$ if and only if $i<j$. So, there are in each block of R a_i cars with destination S_i and b_i cars with destination $T_i, i=1,\ldots,n$, in total there are $\sum_{i=1}^n a_i+\sum_{i=1}^n b_i=\sum_{i=1}^n c_i$ cars in each block of R, thus there are in total $N=n\sum_{i=1}^n c_i$ cars in R, as stated above. The first two subblocks in any block contain n cars with T destinations ($b_i>2, 1\leqslant i\leqslant n$) and no car with S destination ($a_i+2\leqslant c_j,\ 1\leqslant i,j\leqslant n$), while the last subblock is made up of n cars of S destinations ($b_i<c_j,\ 1\leqslant i,j\leqslant n$).

Example. Let $a_1=2, a_2=3, a_3=3, b_1=3, b_2=4, b_3=3, c_1=5, c_2=7, c_3=6$. Then the train R is

$$\begin{aligned}&t_1,t_2,t_3-t_1,t_2,t_3-s_2,s_3,t_1,t_2,t_3-s_1,s_2,s_3,t_2-s_1,s_2,s_3|t_1,t_2,t_3-t_1,t_2,t_3-t_1,t_2,t_3\\&t_2-s_2,s_3-s_1,s_2,s_3-s_1,s_2,s_3|t_1,t_2,t_3-t_1,t_2,t_3\\&-t_1,t_2,t_3-s_2,s_3,t_2-s_1,s_2,s_3-s_1,s_2,s_3.\end{aligned}$$

For the readers convenience, we have separated blocks of R by the symbol $|$, the subblocks inside a block are separated by the symbol $-$. By $s_i(t_i)$ we mean a car with the destination $S_i(T_i)$. Hence, the first car of R has destination T_1, followed by a car of destination T_2, then a car of destination T_3, followed by another car of destination T_1, etc. Formally, the train R consists of $N=3(5+7+6)=54$ cars, i.e., the train R is a partition of the set $\{1,\ldots,54\}$ into six parts, $T_1=\{1,4,9,\ldots,43\}, T_2=\{2,5,10,\ldots,38,41,44,48\}, T_3=\{3,6,11,\ldots,45\},\ldots,S_3=\{8,14,18,\ldots,36,47,51,54\}$.

To show that the Train Marshalling Problem is NP-complete we prove the following statement.

Main Claim. *$K(R)\leqslant 2n-1$ if and only if $a_1,\ldots,a_n,b_1,\ldots b_n,c_1,\ldots,c_n$ have a numerical matching, i.e., the conditions of NMTS are satisfied.*

$\Leftarrow$: Suppose we have found permutations π_1 and π_2 satisfying the conditions of NMTS. We need to show that $K(R)\leqslant 2n-1$, i.e., that there is an order O of the $2n$ destinations so that the sequence

$J=1,2,\ldots,N,1,2,\ldots,N,\ldots,1,2,\ldots,N$, where the interval (round) $1,2,\ldots,N$ is repeated $2n-1$ times contains all cars of the first destination in the order O, followed by all cars of the second destination in O, etc.

As the order O we take the following sequence.

$$O = T_{\pi_2(n)}, S_{\pi_1(n)}, T_{\pi_2(n-1)}, S_{\pi_1(n-1)}, \dots T_{\pi_2(1)}, S_{\pi_1(1)}.$$

Always choose as the first car of $T_{\pi_2(i)}$ on J the car in $D_1^{(i+1) \bmod n}$. Then the car of $T_{\pi_2(i)}$ placed last is in $D^i_{b_{\pi_2(i)}}$.

Choose as the first car of $S_{\pi_1(i)}$ on J one in $D^i_{c_i - a_{\pi_1(i)}+1}$. Note that the car of $S_{\pi_1(i)}$ placed last is in $D^{(i-1) \bmod n}_{c_{(i-1) \bmod n}}$.

Example. Using our previous example, take the sequence $1,2,\dots,54,1,2,\dots,54,\dots,1,2,\dots,54$, where the interval $1,2,\dots,54$ is repeated 5 times. As $a_i + b_i = c_i, i = 1,2,3$, we have $\pi_1(i) = \pi_2(i) = i$. Hence, place the cars (numbers) of T_3 at the beginning of the sequence, first the number 3, the last the number 45. They will be followed by cars (numbers) from S_3, starting with 47, then 51,54, still in the first round of the sequence. The other numbers from S_3 spill over the second round of the sequence, starting with 8, then $14, 18, \dots, 36$. They are followed, in the second round, by cars from T_2 starting with 38, then 41,44, and 48. The other cars from T_2 come at the beginning of the third round, etc.

In general, the cars of the destination $T_{\pi_2(n)}$ will come at the beginning of the first round, the last one being from $D^n_{\pi_2(n)}$. At the end of the first round we place all cars from the destination $S_{\pi_1(n)}$ which are in the block X_n, the first one from $D^n_{c_n - a_{\pi_1(n)}+1}$. This is possible as $a_{\pi_1(n)} + b_{\pi_2(n)} = c_n$, i.e., the last car of $T_{\pi_2(n)}$ is in a subblock preceding the subblock of the first car of $S_{\pi_1(n)}$. The remaining cars of $S_{\pi_1(n)}$ go to the beginning of the second round. The last one of them is in the block X_{n-1}. At the end of the second round we place cars with the destination $T_{\pi_2(n-1)}$ which are in the block X_n. The other cars of the destination are placed at the beginning of the third round, etc. In general, if any destination M has been placed and the last placed car of M is in D^i_j then the first placed car of the next destination in the order O is in that $D^{i'}_{j'}$ which follows D^i_j immediately, i.e., $D^{i'}_{j'} = D^i_{j+1}$, or, in case $j = c_i$, $D^{i'}_{j'} = D^{i+1}_1$.

Moreover, all cars from the first destination are placed in the first round; for $i = 2,\dots,2n-1$, the cars from the ith destination in the sequence $O = T_{\pi_2(n)}, S_{\pi_1(n)}, T_{\pi_2(n-1)}, S_{\pi_1(n-1)}, \dots T_{\pi_2(1)}, S_{\pi_1(1)}$ are placed at the end of the $(i-1)$th round and at the beginning of the ith round. As the last car of the $(2n-1)$th destination in $O, T_{\pi_2(1)}$, is in $D^1_{b_{\pi_2(1)}}$ all cars of the last destination in O, $S_{\pi_1(1)}$, can be placed into the same $(2n-1)$th round and the proof of this part is complete.

$\Rightarrow$: Suppose $K(R) \leqslant 2n-1$. Consider an arrangement of cars of R on the sequence $J = 1,2,\dots,N,1,2,\dots,N,\dots,1,2,\dots,N$ (N being the total number of cars) with $2n-1$ rounds. For each destination M let f_M be the car of destination M being placed first and l_M be the car of destination M being placed last in J.

Let J_M be the interval on J induced by the sequence that starts in f_M and ends in l_M.

Each number m of $I_N = \{1,\dots,N\}$ occurs $2n-1$ times in J. We will say that m is covered by a destination M if at least one of the $2n-1$ occurrences of m in J belongs to the interval J_M. We will use the expression "a car c covered by a destination T" to mean that a number on J which equals the number of the car c in R is covered by T. To illustrate this definition, we go back to our example and use the arrangement given in part $\Leftarrow$ of the proof.

We have $J_{T_3}=[3,45]$, $J_{S_3}=[47,54]\cup[1,36]$. Thus, number 4 is covered by both destinations T_3 and S_3 as J_{T_3} covers number 4 in the first round of J, and J_{S_3} covers number 4 in the second round of J; 46 is covered by neither of them as none of 5 occurrences of number 46 in J belongs to either J_{T_3} or J_{S_3}; and 40 is covered only by T_3. Since we have $2n-1$ rounds in J, each number of I_N occurs in J exactly $2n-1$ times, and hence belongs to at most $2n-1$ intervals J_M (is covered by at most $2n-1$ destinations) as all intervals I_M are pairwise disjoint on J. This fact will be essential in the rest of the proof.

For $i=1,\dots,n$, denote by x_i the number of the car of destination T_n in the subblock D_1^i (this car is the last one in the subblock); denote by y_i the number of the car of destination S_1 in the subblock $D_{c_i}^i$ (this car is the first one in the subblock), respectively. Now, we will characterize destinations not covering at least one of the numbers x_i, y_i. Consider first a T destination, with the first car f_T of T in the subblock D_j^i. If $j\geqslant 2$, then a car of the destination T is also in the subblock preceding D_j^i, i.e., for the last car of T we get $l_T\in D_{j-1}^i$. Thus, the destination (the interval J_T) covers all cars of the train except for cars preceding f_T in the subblock D_j^i and following l_T in the subblock D_{j-1}^i. Therefore, in this case, the destination covers numbers y_t for $t=1,\dots,n$, and covers numbers x_t for $t=1,\dots,n$ with possibly one exception; this happens if $T\neq T_n$ and $j=2$; then T does not cover exactly one of x_t numbers, the number x_i. For $j=1$, the last car l_T of $T=T_k$ is in the subblock $D_{b_k}^{(i-1)\bmod n}$. Hence, T covers all x_t cars (the car $x_{(i-1)\bmod n}$ is covered as $b_k>2$) as well as all cars y_t except for $y_{(i-1)\bmod n}$ which is in the subblock immediately preceding the subblock of f_T since $b_k<c_j$. Consider now an S destinations, $S=S_k$, with $f_{S_k}\in D_j^i$. If $c_i-a_k+1\ <j<c_i$ then the last car of S_k belongs to the subblock D_{j-1}^i immediately preceding the subblock D_j^i which in turn implies that S_k covers all x_t and all y_t cars. For $j=c_i-a_k+1$, we have $l_{S_k}\in D_{c_{(i-1)\bmod n}}^{(i-1)\bmod n}$, hence S_k covers all x_t and all y_t cars except of the car x_i. For $j=c_i$, l_{S_k} belongs to the subblock $D_{c_i-1}^i$ immediately preceding the subblock $D_{c_i}^i$ ($a_i\geqslant 2$), all x_i cars are covered by S_k ($c_i\geqslant a_j+2$), and, if $S_k\neq S_1$, the only y_i car which is not covered by S_k is the first car of the subblock. Clearly, S_1 covers all cars y_i, as they are all of destination S_1. Thus, any destination covers all but at most one of the cars x_t, y_t. Summarizing the above analysis we get:

For $i=1,\dots,n$, the car x_i is not covered only by either

(i) any T destination, $T\neq T_n$, with f_T in D_2^i or by

(ii) any S_j destination with f_{S_j} in $D_{c_i-a_j+1}^i$.

Similarly, the car y_i is not covered only by

(iii) any T destination with f_T in $D_1^{(i+1)\bmod n}$ or by

(iv) any S destination, $S\neq S_1$, with f_S in $D_{c_i}^i$.

As mentioned before, any destination of a type (i)–(iv) does not cover exactly one among the $2n$ cars $Y=\{x_i,y_i,i=1,\dots,n\}$. So, each destination has to be of a type (i)–(iv). Consequently, each car of Y is not covered by exactly one destination.

Assume that, for some i, the only destination not covering x_i is of type (i). This assumption leads to a contradiction as then the last car C in the subblock D_2^i would be covered by all destinations. Indeed, from all destinations of a type (i)–(iv) only an S destination of type (ii) with f_S in $D^i_{c_l-a_j+1}$ would not cover C. (A T destination of type (iii) with f_T in D_1^{i+1} covers C since $b_j > 2$, all S destinations of type (iv) cover C as $c_i \geqslant a_i + 2, a_i \geqslant 2$, and a T destination of type (i) covers C as C is the last car in the subblok D_2^i). However, there cannot be an S destination with f_S in $D^i_{c_l-a_j+1}$because this destination would also not cover x_i. This would, in turn, imply that there are two destinations not covering x_i contradicting the fact that each car from Y is not covered by exactly one destination.

Thus there is no T destination of type (i), all of them are of type (iii) which in turn implies that all S destinations have to be of type (ii), otherwise some car of Y would not be covered by two destinations, that is, some car of Y would be covered by all $2n$ destinations, a contradiction.

Hence, for each i, there is exactly one T destination with f_T in the block X_i, and exactly one S destination with f_S in X_i. Consider a block X_i for a fixed i. Let T_k be the T destination with $f_{T_k} \in X_{(i+1) \bmod n}$, and S_j be the S destination with $f_{S_j} \in X_i$. T_k and S_j are the only two destinations which do not cover cars in subblocks of X_i. Thus, $l_{T_k} < f_{S_j}$, otherwise all cars of R between f_{S_k} and l_{T_k} inclusive would be covered by all $2n$ destinations. Moreover, since in each subblock cars with S destinations are placed in front of T destination cars, l_{T_k} is in a subblock which precedes the subblock containing f_{S_k}. As the destination T_k does not cover the last $c_i - b_k$ subblocks in X_i, the destination S_j does not cover the first $c_i - a_j$ subblocks in X_i, we have $c_i - b_k + c_i - a_j \geqslant c_i$, i.e., $c_i \geqslant a_j + b_k$. If we set $\pi_1(i) = j, \pi_2(i) = k$, then we get, for $i = 1, \ldots, n$, $c_i \geqslant a_{\pi_1(i)} + b_{\pi_2(i)}$.

However, $\sum_{i=1}^n a_i + \sum_{i=1}^n b_i = \sum_{i=1}^n c_i$ and so we have to have equality in the previous inequality for all $i = 1, \ldots, n$.

The NMTS follows. $\square$

3. An upper bound

Since the Train Marshalling Problem is NP-complete, in this section we provide a general upper bound on $K(S)$ in terms of n, the total number of cars in a train. For that purpose, we set the number K_n to be the maximum of $K(S)$, where S ranges over all partitions of the set $\{1, \ldots, n\}$. The main result of this part of the paper is the following statement:

Theorem. $K_n = \lceil n/4 + 1/2 \rceil$

We will prove by induction on n that $K_n \leqslant \lceil n/4 + 1/2 \rceil$ and produce a partition S of $\{1, 2, \ldots, n\}$ such that $K(S) = \lceil n/4 + 1/2 \rceil$.

First, we introduce some more notation and state several observations.

We assume that S is a partition of I_n, $S=\{S(1),S(2),\ldots,S(s)\}$. By S_i we denote the element of S containing the number i. If some number will be denoted by x_i, y_i, z_i, etc. then it will be a number from the same element of S as the number i. So, $S_i=\{i,x_i,y_i,\ldots\}$. If we want to show that $K(S)=t$ then sometimes we will list all elements in the individual t rounds. In such a case the symbol $*$ indicates that we start a new round. For example, if $S=\{\{1,5\},\{2,6\},\{3,7\},\{4,8\}\}$, then to show that $K(S)=3$ we write:

$$156*2378*4.$$

We will frequently make use of the following simple observation. Let T^{-1} be a train obtained from a train T by reversing the order of its cars. This means that the car which was the first in T will be the last car in T^{-1}, etc. It is easy to see that

$$K(T)=K(T^{-1}). \tag{$\dagger$}$$

In the terminology of intervals, for a partition $S=\{S(1),\ldots S(s)\}$, $S(j)=\{a_{j_1},\ldots,a_{j_r}\}$ an inverse partition $S^{-1}=\{S'(1),\ldots,S'(s)\}$ is given by $S'(j)=\{n-a_{j_1}+1,\ldots,n-a_{j_r}+1\}$. Now (†) translates to $K(S)=K(S^{-1})$.

Let $T=\{a_1,\ldots,a_n\}$, and $T'=\{a_{i_1},\ldots,a_{i_n}\}$ be trains where $i_1<\cdots<i_n$ is an increasing sequence of indices, and two cars a_j, a_k of T are of the same destination iff the cars a_{i_j}, a_{i_k} of T' have the same destination. Then clearly $K(T)=K(T')$. We will often utilize this trivial statement in the following form.

Let $S'=\{S(i_1),\ldots,S(i_r)\}$ be a subset of the partition S, where $|S(i_1)\cup\cdots\cup S(i_r)|=t$. Then

$$K(S)\leqslant K(S')+K(S-S')\leqslant K(S')+K_{n-t}.$$

We start with some simple observations.

Observation 1. *Let S contain a destination with r cars, or r destinations having a single car each. Then $K(S)\leqslant 1+K_{n-r}$.*

Proof. To get an arrangement of all cars into $1+K_{n-r}$ rounds, it is sufficient to place cars of the destination with r cars (cars from the r destinations with a single car) into the first round and then use an arrangement of the remaining $n-r$ cars into K_{n-r} rounds. □

Observation 2. *Let S' be a partition obtained from S by placing all cars from single car destination into one new destination. Then $K(S)\leqslant K(S')$.*

Proof. Clearly, an arrangement of cars of S' is also an arrangement of cars of S. □

If all numbers of a destination $S(i)$ are smaller than all numbers of a destination $S(j)$ we say that $S(i)$ and $S(j)$ do not overlap.

Observation 3. *If* $|S(i)| + |S(j)| = r$ *and* $S(i)$ *and* $S(j)$ *do not overlap then* $K(S) \leqslant 1 + K_{n-r}$.

Proof. In this case it is possible to place all cars of $S(i)$ and $S(j)$ into the first round and the statement follows. □

Now, we are ready to prove by induction that $K_n \leqslant \lceil n/4 + \frac{1}{2} \rceil$ for all $n \geqslant 1$, i.e., that for all partitions S of $\{1,\dots,n\}, K(S) \leqslant \lceil n/4 + \frac{1}{2} \rceil$. The statement is obvious for $n \leqslant 4$. Clearly, for all n, $K_n \leqslant K_{n+1}$. As $\lceil n/4 + \frac{1}{2} \rceil = \lceil (n+1)/4 + \frac{1}{2} \rceil$ for n odd, it is sufficient to prove the statement for all even numbers n (actually we will use the fact that n is even only in one subcase of the proof of Case 4). Now, let $S = \{S(1), S(2), \dots, S(s)\}$ be a partition of $\{1,\dots,n\}$, $n \geqslant 5$. In what follows, we assume that at most one destination contains a single car. In the other case we would substitute S with S' as in Observation 2 and prove the statement for S'. Moreover, we also assume that $2 \leqslant |S(i)| \leqslant 3$ for all other destinations. In the other case Observation 1 and the induction hypothesis gives the desired statement. Finally, we also assume that any two destinations with at least four cars in total overlap, otherwise Observation 3 and the induction hypothesis would take care of the case.

We will make use of the following auxiliary statement.

Lemma. *Let* $S(i_1), \dots, S(i_k)$ *be destinations of* S *such that* $|S(i_1)| + \cdots + |S(i_k)| = t$, *and all the numbers of the interval* $\{1,\dots,j\}$ *belong to* $U = S(i_1) \cup \cdots \cup S(i_k)$. *If it is possible to arrange all numbers from* U *into* k *rounds so that the last* (k*th*) *round contains only numbers from the interval* $\{1,\dots,j\}$, *then* $K(S) \leqslant k - 1 + K_{n-t}$.

Proof. Suppose that the only numbers in the last (kth) round of the arrangement A of the elements from U are from the interval $\{1,\dots,j\}$. Then we can arrange all numbers by starting with the first $k-1$ rounds of A, the kth round will be made up by numbers from $\{1,\dots,j\}$ belonging to the kth round of A followed by the numbers from the first round of an arrangement A' of the numbers not in U into K_{n-t} rounds (note that they are all bigger than j), and then appending the other rounds of A'. In total we have $k - 1 + K_{n-t}$ rounds. □

When applying the above Lemma, for the sake of convenience, we will not explicitly exclude partitions S with $t = n$ (partitions S having exactly k parts). For such partitions S Lemma would yield $K(S) \leqslant k - 1 + K_0$. Therefore, we define $K_0 = 1$, and then Lemma gives only a trivial bound $K(S) \leqslant k$.

We will distinguish cases with respect to the cardinality of S_1 and S_2.

Case 1: *Let* $|S_1| = |S_2| = 2$. *Then* $K(S) \leqslant 1 + K_{n-4} \leqslant \lceil n/4 + \frac{1}{2} \rceil$.

Proof. Clearly, $S_1 = \{1, x_1\}$ differs from $S_2 = \{2, x_2\}$, otherwise S_1 would not overlap the other destinations. If $x_2 > x_1$, then the arrangement $1, x_1, x_2, *, 2$, otherwise the arrangement $2, x_2, x_1, *, 1$ satisfies the assumptions of Lemma, and we are done. □

Case 2: *If one of* S_1, S_2 *is of cardinality* 1, *then* $K(S) \leqslant 1 + K_{n-4} \leqslant \lceil n/4 + \frac{1}{2} \rceil$.

Proof. Suppose first $|S_1| = 1$. If at least one destination is of cardinality 3 then it does not overlap with S_1, and the statement follows by Observation 3. When there is no destination of cardinality 3 we have $|S_{n-1}| = |S_n| = 2$, because by $n \geqslant 5$ neither set is S_1 and there is at most one single car destination. Then Case 1 together with (†) takes care of this case. Now let $|S_2| = 1$. If $S_1 = \{1, x_1, y_1\}$, then the arrangement $2, x_1, y_1, *, 1$ satisfies Lemma. Otherwise, $|S_1| = 2$, because there is only one destination with a single car. If there is a destination with 3 cars it does not overlap S_2, and the statement follows by Observation 3. Otherwise, we again have $|S_{n-1}| = |S_n| = 2$, and the claim follows as before. □

Case 3: *Let* $|S_1| = |S_2| = 3$. *Then either* $K(S) \leqslant 1 + K_{n-4} \leqslant \lceil n/4 + \frac{1}{2} \rceil$ or $K(S) \leqslant 2 + K_{n-8} \leqslant \lceil n/4 + \frac{1}{2} \rceil$.

Proof. Let $S_1 = \{1 < x_1 < y_1\}, S_2 = \{2 < x_2 < y_2\}$. First assume $S_1 \neq S_2$. If $|S_3| = 1$ then it is sufficient to apply Lemma to S_1 and S_3 with the arrangement $3, x_1, y_1, *, 1$. Assume that $|S_3| = 3$, $S_3 = \{3 < x_3 < y_3\}$, and $S_1 \neq S_3 \neq S_2$, the other cases will be treated later. Denote by M the largest number in $U = S_1 \cup S_2 \cup S_3$. Let $j \in \{1, 2, 3\}$ be the index of the set S_j containing M, thus $S_j = \{j, x_j, M\}$. The arrangements of cars of U into three rounds given below satisfy the assumptions of Lemma, which in turn implies $K(S) \leqslant 2 + K_{n-9} \leqslant \lceil n/4 + \frac{1}{2} \rceil$.

Let $\{i, k\} = \{1, 2, 3\} - \{j\}$, and let the naming be such that $y_i < y_k$. If $x_k < x_j$, then $i, x_i, y_i, y_k * k, x_k, x_j, M * j$ is the desired arrangement. Otherwise, $x_j < x_k$, and $i, x_i, y_i, M * j, x_j, x_k, y_k * k$ is the desired arrangement.

Next, suppose that $|S_3| = 2$. Then we add to S_3 a dummy car which comes to the end of the train, i.e., $S_3 = \{3, x_3, n+1\}$, denoting the new partition S'. By the same token as before we have $K(S) \leqslant K(S') \leqslant 2 + K_{n+1-9} \leqslant 2 + K_{n-8} \leqslant \lceil n/4 + \frac{1}{2} \rceil$. At the end of the proof we note that in the case $S_1 = S_2 = \{1, 2, x\}$ and $|S_3| < 3$ we can always apply Lemma to S_1 and S_3. If $S_3 = \{3\}$, then $1, 2, x * 3$ is the desired arrangement; if $|S_3| = 2, S_3 = \{3, y\}$, then the desired arrangement is $1, 2, x, y * 3$ for $x < y$, and $3, y, x * 1, 2$ for $y < x$. If $S_1 = S_2$ and $|S_3| = 3$ or one of S_1, S_2 equals S_3 then we take $S_4, S_5, \ldots$ until we get the third different destination and apply the above argument. □

Case 4: *Let* n *be an even number. If* $|S_1| + |S_2| = 5$ *then either*
(a) $K(S) \leqslant 1 + K_{n-5} \leqslant \lceil n/4 + \frac{1}{2} \rceil$ or
(b) $K(S) \leqslant 2 + K_{n-8} \leqslant \lceil n/4 + \frac{1}{2} \rceil$.

Proof. In this proof we will denote the elements of destinations S_1, S_2 by $\{a_1 < a_2\}$, $\{b_1 < b_2 < b_3\}$, $\{a_1, b_1\} = \{1, 2\}$ in order not to have to distinguish whether the destination S_1 is of cardinality 2 or 3. Further, consider a partition S^{-1} inverse to the partition S. From (†) we know that $K(S) = K(S^{-1})$. If one of Cases 1–3 applies to S_{n-1} and S_n we are done. Therefore we assume that $|S_{n-1}| + |S_n| = 5$, and denote the cars of these destinations by $\{u_1 < u_2\}, \{v_1 < v_2 < v_3\}$, where $\{u_2, v_3\} = \{n-1, n\}$. We need to consider only three possibilities. If $a_2 < b_2 < b_3$, then the arrangement $a_1, a_2, b_2, b_3 * b_1$ satisfies the assumption of Lemma; if $b_2 < b_3 < a_2$, then the arrangement $b_1, b_2, b_3, a_2 * a_1$, satisfies the assumptions of Lemma. In either case we get $K(S) \leqslant 1 + K_{n-5} \leqslant \lceil n/4 + \frac{1}{2} \rceil$. If

$$b_2 < a_2 < b_3, \tag{‡}$$

we are not able to apply Lemma to S_1 and S_2. Now we consider two cases.

Case 4a: *There is a destination different from* S_1, S_2, *having* 3 *cars, say* $S(j) = \{c_1 < c_2 < c_3\}$. *Let* $1, 2, d_3 < d_4 < \ldots < d_8$ *be the numbers of* U, *the union of* S_1, S_2, S_3 *in the increasing order. Because of* (‡) *and the fact that all destinations overlap, we get for* a_2, *that* $a_2 = d_i$, *where* $i \in \{5, 6, 7\}$ (*i.e.*, a_2 *is at least the fifth largest number in* U *but not the largest one*). *The following arrangements satisfy the assumptions of Lemma, hence* $K(S) \leqslant 2 + K_{n-8} \leqslant \lceil n/4 + \frac{1}{2} \rceil$.

If $a_2 = d_7$, *the arrangement is* $b_1, b_2, b_3 * c_1, c_2, c_3, a_2 * a_1$.

If $a_2 = d_6$ *and* $b_3 > c_3$, *the arrangement is* $c_1, c_2, c_3, b_3 * b_1, b_2, a_2 * a_1$. *If* $a_2 = d_6$ *and* $c_3 > b_3$, *the arrangement is* $b_1, b_2, b_3, c_3 * c_1, c_2, a_2 * a_1$.

If $a_2 = d_5$ *then both* b_2 *and* c_1 *are strictly smaller than* a_2 *which is strictly smaller than any of* b_3, c_2, c_3. *We again consider three possibilities. For the subcases* $b_3 < c_2 < c_3$ *and* $c_2 < c_3 < b_3$ *the arrangements* $b_1, b_2, b_3, c_2, c_3 * c_1, a_2 * a_1$ *and* $c_1, c_2, c_3, b_3 * b_1, b_2, a_2 * a_1$, *respectively, satisfy Lemma, and we get* $K(S) \leqslant 2 + K_{n-8} \leqslant \lceil n/4 + \frac{1}{2} \rceil$. *The last subcase, when*

$$d_5 = a_2 < c_2 < b_3 < c_3 \tag{††}$$

is the most difficult.

Note that in this case $\{b_1, b_2, b_3\} \neq \{v_1, v_2, v_3\}$, otherwise $b_3 > c_3$. Therefore now we may take $S(j) = \{v_1, v_2, v_3\}$. In view of (††) we have $a_2 < v_2 < b_3 < v_3$. Further, if (‡) is not satisfied in S^{-1} for S_{n-1} and S_n a previous case would apply, hence $v_1 < u_1 < v_2$. If $\{a_1, a_2\}$ and $\{u_1, u_2\}$ do not overlap, we would be done by Observation 3, thus $u_1 < a_2$. Finally, (††) translated to S^{-1} for S_{n-1}, S_n, and to $(j) = \{b_1, b_2, b_3\}$ reads as $b_1 < v_1 < b_2 < u_1$. Combining the above inequalities we get that the only case which has not yet been taken care of is the case where the two smallest numbers are a_1, b_1 in arbitrary order, then $v_1 < b_2 < u_1 < a_2 < v_2 < b_3$ and finally v_3, u_2 in any order. However, we can apply Lemma to the destinations $S_1, S_2, \{v_1, v_2, v_3\}$ and the arrangement $a_1, a_2, v_2, v_3 * v_1, b_2, b_3 * b_1$.

Case 4b: $\{b_1, b_2, b_3\}$ *is the only destination with three cars. This implies* $\{b_1, b_2, b_3\} = \{v_1, v_2, v_3\}$. *Since* n *is an even number there has to be a destination with a single car* $\{w\}$. *Again we consider two cases.*

Case 4b1: $\{a_1, a_2\} \neq \{u_1, u_2\}$. it Because of (‡), *and* (‡) *applied to* S^{-1} *we have* $b_2 < a_2 < b_3$, *and* $b_1 < u_1 < b_2$. *In addition,* $b_1 < w < b_3$, *otherwise we would be done by Observation* 3. *With respect to* (†), *we may assume* $b_2 < w$. *Thus, we need to* *consider only two cases*

(i) $w < a_2$, *so the first two cars of the train are* a_1, b_1, *then* $u_1 < b_2 < w < a_2$ *and the cars* b_3, u_2 *form the very end of the train.*

(ii) *Same as case* (i) *with the order of* a_2 *and* w *interchanged.*

We apply Lemma to destinations S_1, S_2, $\{u_1, u_2\}$, and $\{w\}$ with the arrangement

(i) $b_1, b_2, b_3, u_2 * u_1, w, a_2 * a_1$ for $u_2 > b_3$

$u_1, u_2, b_3 * b_1, b_2, w, a_2 * a_1$ for $u_2 < b_3$;

(ii) $a_1, a_2, w, u_2 * u_1, b_2, b_3 * b_1$.

Case 4b2: $\{a_1, a_2\} = \{u_1, u_2\}$. *By* (†) *and* (‡) *it has to be* $b_1 < a_1 < b_2 < a_2 < b_3$, *i.e.*, $\{b_1, b_2, b_3\} = \{1, b_2, n\} = S_1$. *By* (†) *we may assume that* $1 < w < b_2$. *Applying Lemma to the destinations* S_1 and $\{w\}$ *with the arrangement* $w, b_2, n * 1$ *finishes the proof of this case.* □

The proof of the statement $K_n \leqslant \lceil n/4 + \frac{1}{2} \rceil$ for $n \geqslant 1$ is now complete.

Finally, we present an example of a partition S of I_n with $K(S) = \lceil n/4 + \frac{1}{2} \rceil$.

Let $k = \lfloor n/2 \rfloor$ and $S = \{S_1, \ldots, S_k\}$, where $S_i = \{i, k+i\}$, $i = 1, \ldots, k$. For n odd we add the number n to S_1. Let $J = 1, 2, \ldots, n, 1, 2, \ldots, n, \ldots, 1, 2, \ldots, n$ be a sequence with $K(S)$ rounds and A is an arrangement of numbers of cars of S in J. Then any round of J contains at most 4 cars and, moreover, the first round contains at most 3 cars. Hence, $K(S) \geqslant \lceil (n-3)/4 + 1 \rceil = \lceil n/4 + \frac{1}{2} \rceil$ for $n \neq 3 \bmod 4$. To show the equality for the case $n \equiv 3 \bmod 4$ we need a slightly finer argument. We show that in this case there are at least two rounds of J with at most 3 cars. If the cars of the destination S_1 are placed to the first round then the first and also the second round contain at most 3 cars. Otherwise, the first round and the round where the car $k+1$ of the destination S_1 is placed contain at most 3 cars. Hence, $K(S) \geqslant \lceil (n-6)/4 + 2 \rceil = \lceil n/4 + \frac{1}{2} \rceil$ also in this case. □

4. Concluding remarks

We have shown that if the only piece of information which is available to the dispatcher is the total number n of cars in a train then he/she has to assume that he/she needs for the train at least K_n auxiliary rails. We have also shown that this bound is the best possible. However, in most cases the bound is too rough. As the dispatcher knows that the TMP is NP-complete he/she would appreciate some better bound. The cost he/she has to pay is introducing another parameter. It is reasonable to introduce a parameter which is easily available to the dispatcher. The minimum number of cars in a destination can play this role. More formally, let $S = \{S_1, \ldots, S_t\}$ be a partition of $\{1, \ldots, n\}$. Set $m = \min |S_i|$, $i = 1, \ldots, t$. Denote by $K(n, m) = \max K(S)$, where the maximum is taken over all trains with the total of n cars such that each destination contains at least m cars. We believe that the following is true:

Conjecture. $K(n, m) \leqslant \lceil (m-1)d^2/n + 1/m \rceil$, *where* $d = \lfloor n/m \rfloor$.

To show that if the conjecture is correct then it is the best possible we present a partition $S=\{S_1,\ldots,S_d\}$ of $\{1,\ldots,n\}$ for which $K(S)\geqslant\lceil(m-1)d^2/n+1/m\rceil$.

Put $S_i=\{i,i+d,\ldots,i+(m-1)d\}$, $i=1,\ldots,d$. If n is not divisible by m then the numbers which have not been put in some S_i yet are placed in S_1. Each destination covers on the sequence $J=1,2,\ldots,n,1,2,\ldots,n,\ldots,1,2,\ldots,n$ an interval containing at least $l=1+(m-1)d$ numbers, while the destination S_1, when n is not divisible by m, covers an interval with at least $l+1$ numbers. As we have d destinations the bound follows.

The above result also shows that if $d<\sqrt{n}$ (in this case $m>\sqrt{n}$) then there is a partition with d destinations for which we cannot get anything better than $K(S)=d$.

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